Advanced optimization

2019

About the class

1 assignment:
data fitting

1 exam:
practical example

2 bonus exercises
(10%)

Derivative

$$f(x)=k\in\mathbb{R}\Rightarrow f'(x)=0$$

$$f(x)=x\Rightarrow f'(x)=1$$

$$f(x)=x^k\Rightarrow f'(x)=kx^{k-1}$$

$$f(x)=\frac{1}{x}\Rightarrow f'(x)=-\frac{1}{x^2}$$

$$f(x)=\sqrt{x}\Rightarrow f'(x)=\frac{1}{2\sqrt{x}}$$

$$f(x)=\ln x\Rightarrow f'(x)=\frac{1}{x}$$

$$f(x)=\log_a x\Rightarrow f'(x)=\frac{1}{x\ln a}$$

$$f(x)=e^x\Rightarrow f'(x)=e^x$$

$$f(x)=a^x\Rightarrow f'(x)=a^x\ln a$$

$$f(x)=\sin x\Rightarrow f'(x)=\cos x$$

$$f(x)=\cos x\Rightarrow f'(x)=-\sin x$$

$$f(x)=\tan x\Rightarrow f'(x)=\sec^2 x=1+\tan^2 x$$

$$f(x)=\arcsin x\Rightarrow f'(x)=\frac{1}{\sqrt{1-x^2}}$$

$$f(x)=\arctan x\Rightarrow f'(x)=\frac{1}{1+x^2}$$

Derivative of combined functions

$$\frac{d}{dx}\left(f(x) * g(x)\right) = f'(x)g(x) + g'(x)f(x)$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - g'(x)f(x)}{\left(g(x)\right)^2}$$

Derivation of composite functions

$$h(g(x))^{'} = \underline{h^{'}(g(x))}.\underline{g^{'}(x)}$$

Example:

$$f^{'}(x) = \sqrt{(x^2+1)}$$

$$h(x) = \sqrt{.} \qquad g(x) = (.)^{\frac{1}{2}}$$

$$\underline{h^{'}(g(x)) = \frac{1}{2}(x^2+1)^{-\frac{1}{2}}} \qquad \underline{g^{'}(x) = 2x}$$

$$f^{'}(x) = \frac{1}{2}(x^2+1)^{-\frac{1}{2}}.2x$$

General Form of Optimization Problems

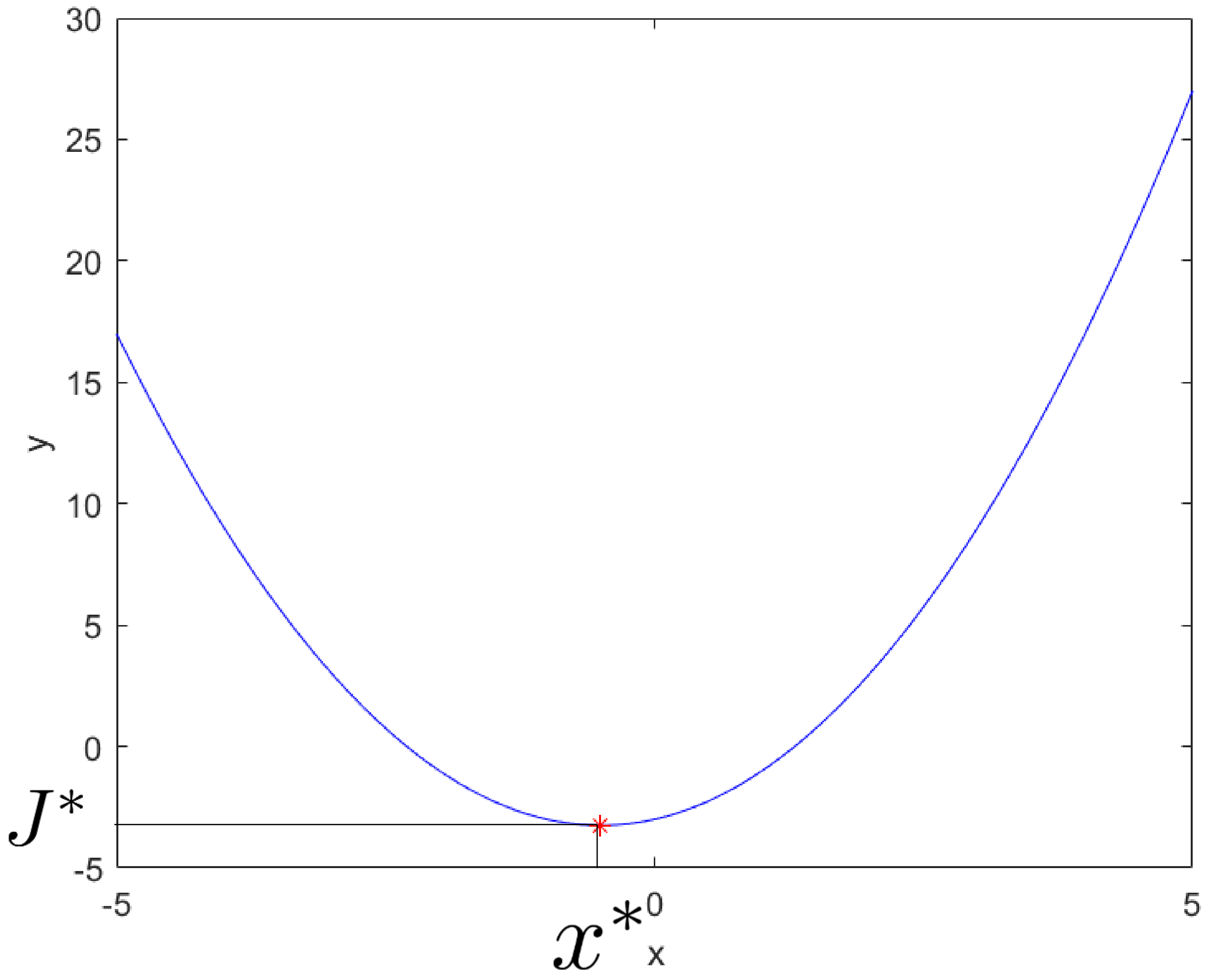
$$\begin{aligned} J^* = \min \quad & f(x) \\ \text{s.t.} \quad & x \in \mathbb{X} \end{aligned}$$

Value of objective function

Objective function

Constrains

General Form of Optimization Problems



General Form of Optimization Problems

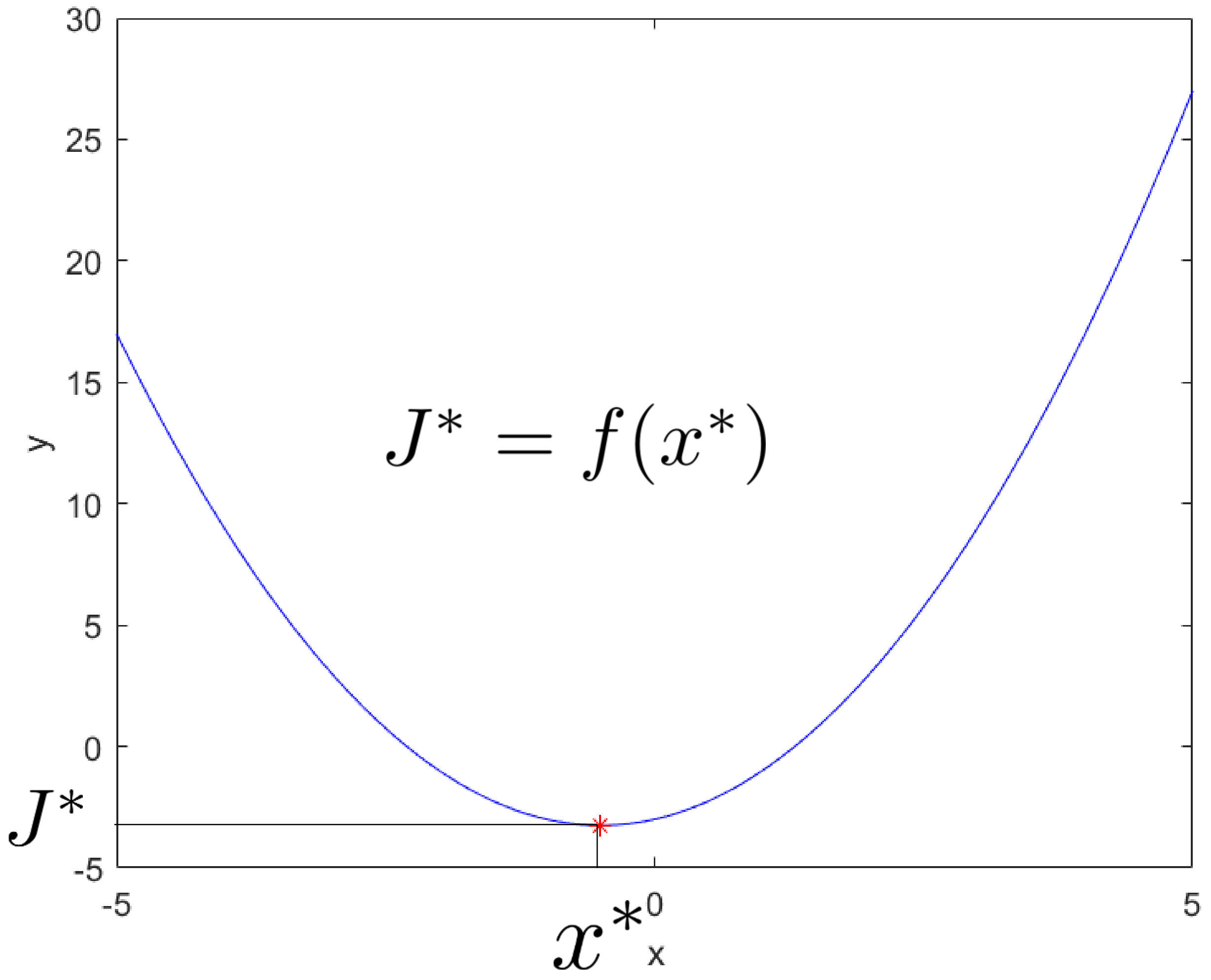
$$x^* = \arg\min \quad f(x)$$
$$\text{s.t.} \quad x \in \mathbb{X}$$

Optimal solution

Objective function

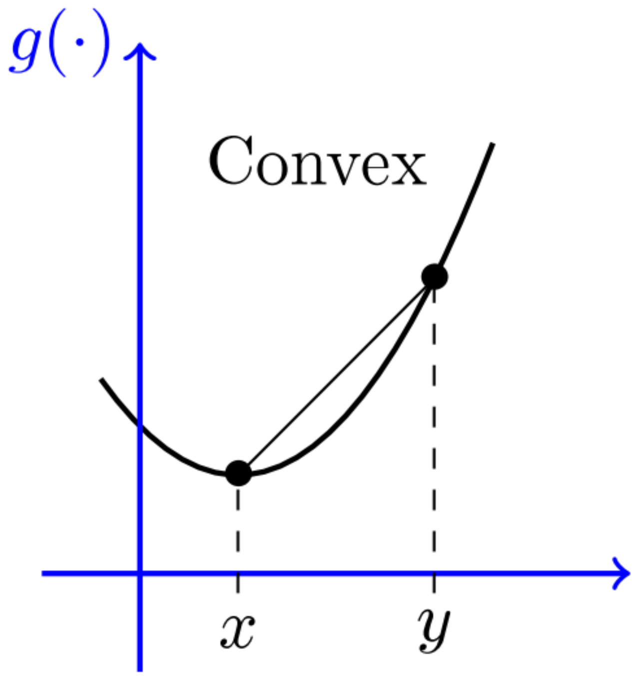
Constrains

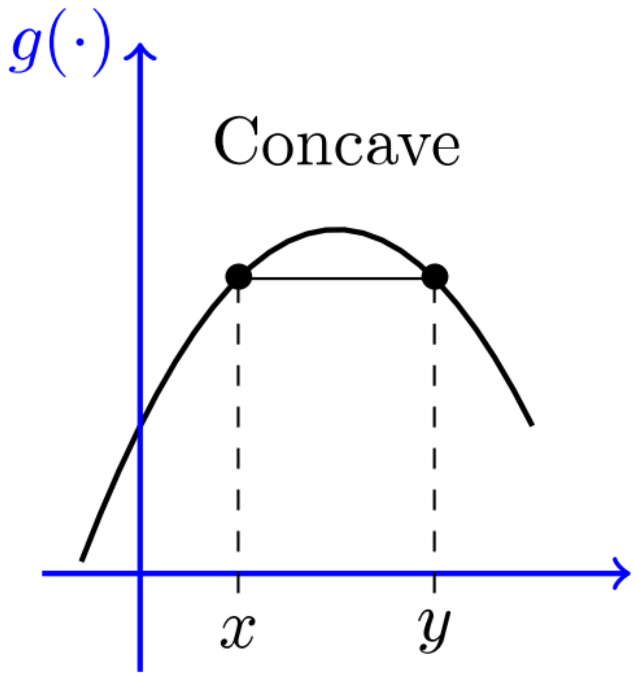
General Form of Optimization Problems



General Form of Optimization Problems

Minimalization vs. maximalization





General Form of Optimization Problems

Minimalization vs. maximalization

$$\begin{aligned} J^* = \max \quad & f(x) \\ \text{s.t.} \quad & x \in \mathbb{X} \end{aligned}$$

$$\begin{aligned} J^* = -\min \quad & f(x) \\ \text{s.t.} \quad & x \in \mathbb{X} \end{aligned}$$

$$\begin{aligned} J^* = \max \quad & x^2 - 3x \\ \text{s.t.} \quad & x \in \mathbb{X} \end{aligned}$$

$$\begin{aligned} J^* = -(x^2 - 3x) \\ \text{s.t.} \quad x \in \mathbb{X} \end{aligned}$$

Unconstrained Optimisation

$$x^* = \arg\min \quad f(x)$$
$$\text{s.t.} \quad x \in \mathbb{X}$$

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Solution:

Analytical (derivates, gradients) – only for convex functions

Minimize given function – Analytical solution

$$\min x^2 + x - 3$$

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1st step:

Derive and set equal to zero $f^{'}(x^*) = 0$

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2nd step:

Solve equation from 1st step

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$$\min x^2 + x - 3$$

1st step:

Derive and set equal to zero $f^{'}(x^*) = 0$

Result: $2x^* + 1 = 0$

2nd step:

Solve equation from 1st step

Result: Minimum, maximum or inflection point $x^* = -\frac{1}{2}$

Minimize given function – Analytical solution

$$\min x^2 + x - 3$$

1st step:

Derive and set equal to zero $f^{'}(x^*) = 0$

Result: $2x^* + 1 = 0$

2nd step:

Solve equation from 1st step

Result: Minimum, maximum or inflection point $x^* = -\frac{1}{2}$

3rd step:

Derive more until you get constant $f^{(n)} = const.$

Result: $f^{''}(x^*) = 2$

Minimize given function – Analytical solution

$$\min x^2 + x - 3$$

1st step:

Derive and set equal to zero $f^{'}(x^*) = 0$

Result: $2x^* + 1 = 0$

2nd step:

Solve equation from 1st step

Result: Minimum, maximum or inflection point $x^* = -\frac{1}{2}$

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Derive more until you get constant $f^{(n)} = const.$

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4th step:

Define if $x^* = -\frac{1}{2}$ is minimum, maximum or inflection point

Minimize given function – Analytical solution

$$\min x^2 + x - 3$$

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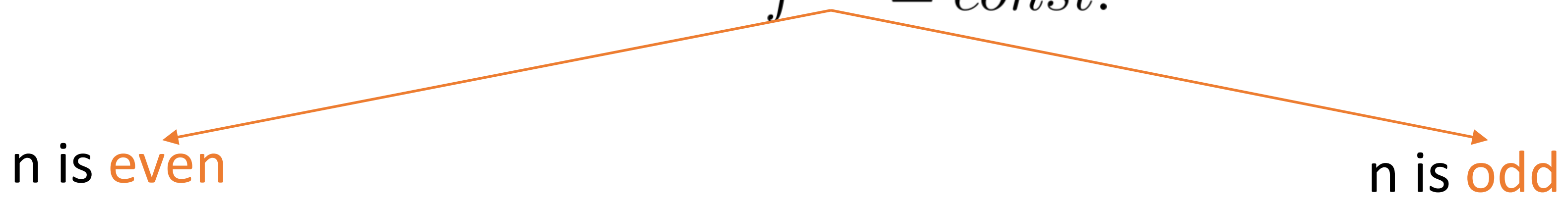

Minimize given function – Analytical solution

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4th step:

Define if $x^* = -\frac{1}{2}$ is minimum, maximum or inflection point

$$f^{(n)} = const.$$

n is even

n is odd

x^* is inflection point

Minimize given function – Analytical solution

$$\min x^2 + x - 3$$

3rd sep:

Derive more until you get constant $\quad f^{(n)} = const.$

Result: $\quad f^{''}(x) = 2$

4th step:

Define if $x^* = -\dfrac{1}{2}$ is minimum, maximum or inflection point

$$f^{(n)} = const.$$

- n is even
  - $const. > 0$
  - $const. < 0$
- n is odd
  - x^* is inflection point

Minimize given function – Analytical solution

$$\min x^2 + x - 3$$

3rd sep:

Derive more until you get constant $f^{(n)} = const.$

Result: $f^{''}(x) = 2$

4th step:

Define if $x^* = -\frac{1}{2}$ is minimum, maximum or inflection point

$$f^{(n)} = const.$$

- n is even
  - $const. > 0$: x^* is minimum
  - $const. < 0$: x^* is maximum
- n is odd
  - x^* is inflection point

Minimize given function – Analytical solution

$$\min x^2 + x - 3$$

4th step:

Define if $x^* = -\frac{1}{2}$ is minimum, maximum or inflection point

Our case: $f^{''}(x) = 2$

Minimize given function – Analytical solution

$$\min x^2 + x - 3$$

4th step:

Define if $x^* = -\frac{1}{2}$ is minimum, maximum or inflection point

Our case: $f''(x) = 2$

↓

n is even

Minimize given function – Analytical solution

$$\min x^2 + x - 3$$

4th step:

Define if $x^* = -\frac{1}{2}$ is minimum, maximum or inflection point

Our case: $f^{''}(x) = 2$

n is even $const. > 0$

Minimize given function – Analytical solution

$$\min x^2 + x - 3$$

4th step:

Define if $x^* = -\frac{1}{2}$ is minimum, maximum or inflection point

Our case: $f''(x) = 2$

n is even $const. > 0$

$x^* = -\frac{1}{2}$ is minimum

Task 1:

Compute and plot optimum of given function in MATLAB

$$\min x^2 + x - 3$$

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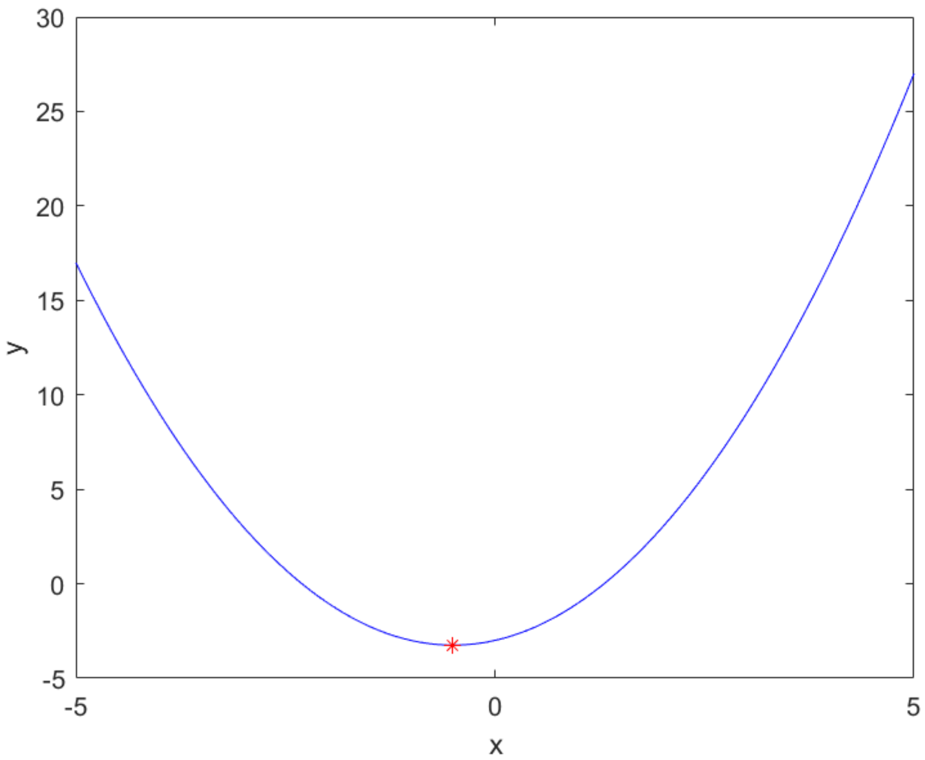

$$x^* = -0.5 \quad J^* = -3.25$$

Task 2:

Compute and plot optimum of given function in MATLAB

$$\min x^2 + 3x + y^2$$

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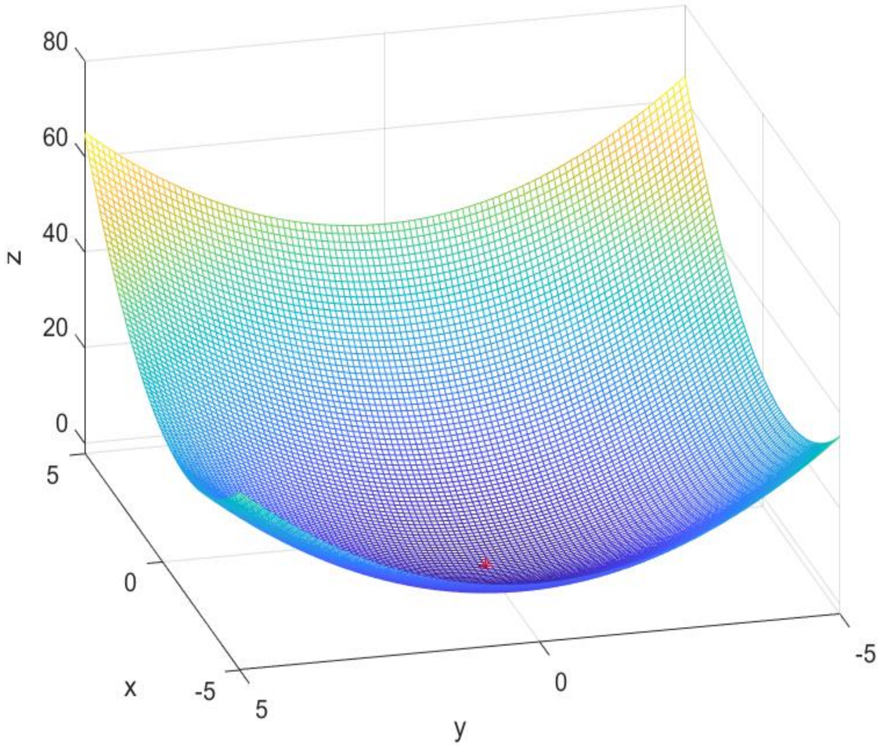
Compute and plot optimum of given function in MATLAB

$$\min x^2 + 3x + y^2$$



$$x^* = -1.5 \quad y^* = 0 \quad J^* = -2.25$$