Control Performance

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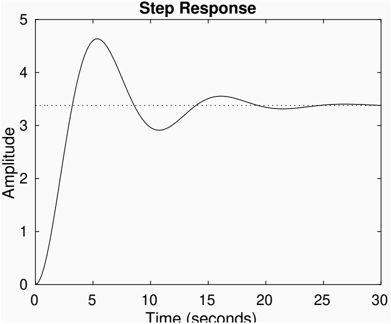
TAR 1

Contents

Quiz

Match the transfer function to the graph of a step response



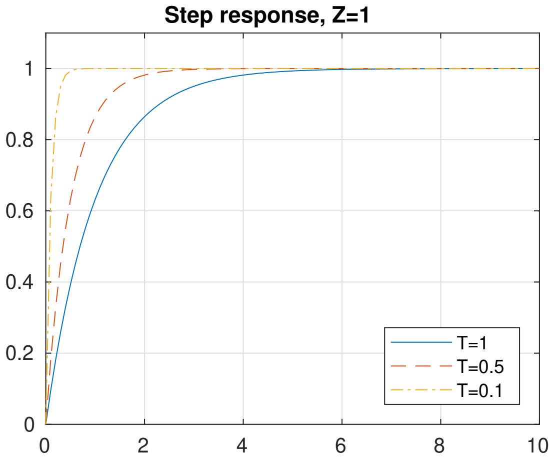
$$G_1 = \frac{3.38}{1.63s + 1}$$

$$G_2 = \frac{-s + 0.03}{2.66s^2 + 3.26s + 1}$$

$$G_3 = \frac{3.38}{1.63s}$$

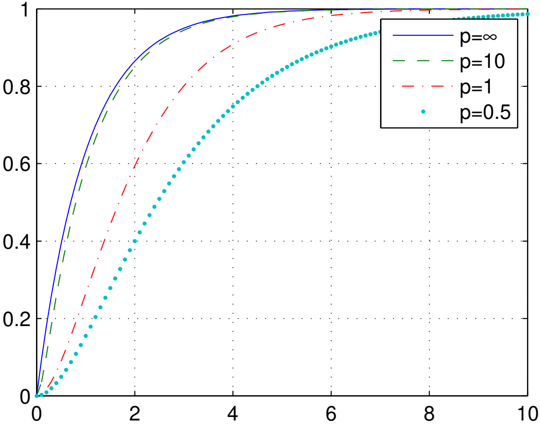
$$G_4 = \frac{3.38}{2.66s^2 + 0.98s + 1}$$

First Order System



$$G(s) = \frac{Z}{Ts+1}, \quad y(t) = Z(1 - e^{-t/T})$$

Additional Pole



$$G(s) = \frac{1}{(s+1)(s/p+1)}, \quad y(t) = 1 - \frac{p}{p-1}e^{-t} + \frac{1}{p-1}e^{-pt}$$

Additional Zero

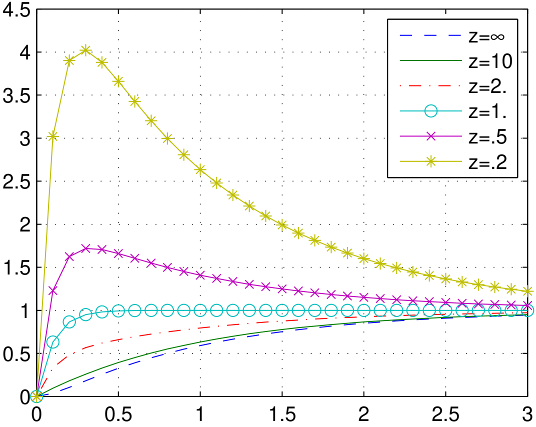
$$G(s) = \frac{s/z+1}{(s/p_1+1)(s/p_2+1)} = (s/z+1)\frac{1}{(s/p_1+1)(s/p_2+1)}$$

$$Y(s) = G(s)U(s) = (s/z+1)\frac{1}{(s/p_1+1)(s/p_2+1)}\frac{1}{s}$$

$$Y_0(s) = G_0(s)U(s) = \frac{1}{(s/p_1+1)(s/p_2+1)}\frac{1}{s}$$

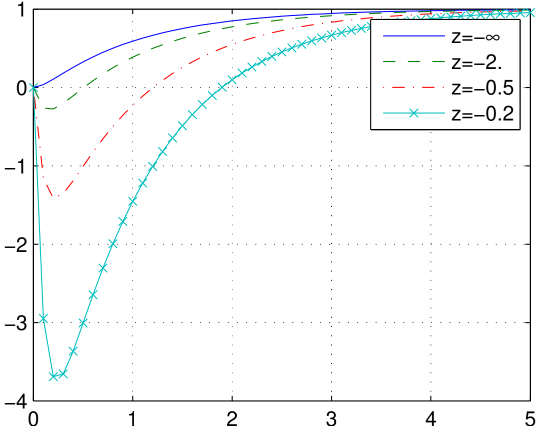
$$y(t) = y_0(t) + \frac{1}{z}\dot{y}_0(t)$$

Additional Zero



$$G(s) = \frac{s/z + 1}{(s+1)(s/10+1)}, \quad y(t) = 1 - \frac{10}{9}\frac{z-1}{z}e^{-t} + \frac{1}{9}\frac{z-10}{z}e^{-10t}$$

Right Half-plane Zero



$$G(s) = \frac{s/z + 1}{(s+1)(s/10+1)}, \quad z < 0$$

Underdamped Second Order System

Transfer function:

$$G = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}, \quad \zeta \in (0, 1)$$

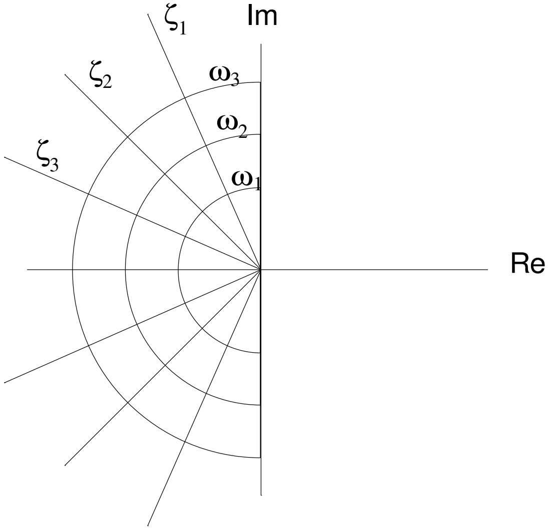
Poles:

$$s_{1,2} = -\zeta\omega_0 \pm j\omega_0 P, \quad P = \sqrt{1 - \zeta^2}$$

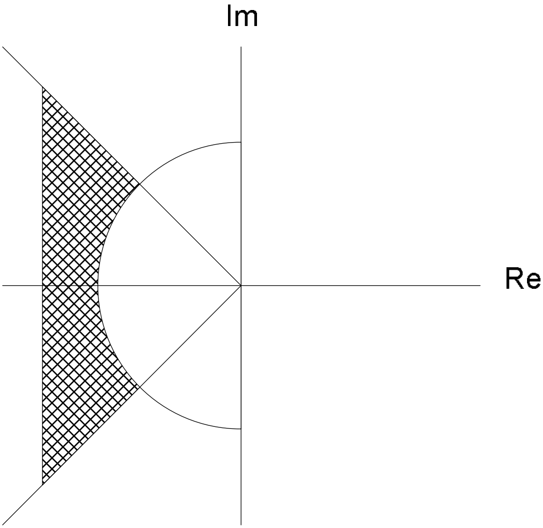
Step response:

$$\begin{aligned} y(t) &= 1 - \frac{1}{P} e^{-\zeta\omega_0 t} \left(P \cos(\omega_0 P t) + \zeta \sin(\omega_0 P t) \right) \\ &= 1 - \frac{1}{P} e^{-\zeta\omega_0 t} \sin(\omega_0 P t + \varphi), \quad \cos\varphi = \zeta \end{aligned}$$

Damping and Frequency



Pole Locations



Second Order System – Time

Maximum overshoot:

$$e_{\max} = e^{-\pi\zeta/P}, \quad \zeta = \frac{|\ln e_{\max}|}{\sqrt{\pi^2 + \ln^2 e_{\max}}}$$

Peak time:

$$t(e_{\max}) = \frac{\pi}{\omega_0 P}$$

Rise time:

$$T_{100} = \frac{1}{\omega_0 P}\left[\pi - \tan^{-1}\left(\frac{P}{\zeta}\right)\right]$$

Useful approximations:

$$T_{\epsilon} \approx \frac{1}{\zeta\omega_0} \ln \frac{1}{\epsilon P}$$

$$T_{0.02} \approx \frac{4}{\zeta\omega_0} \qquad T_{0.05} \approx \frac{3}{\zeta\omega_0}$$

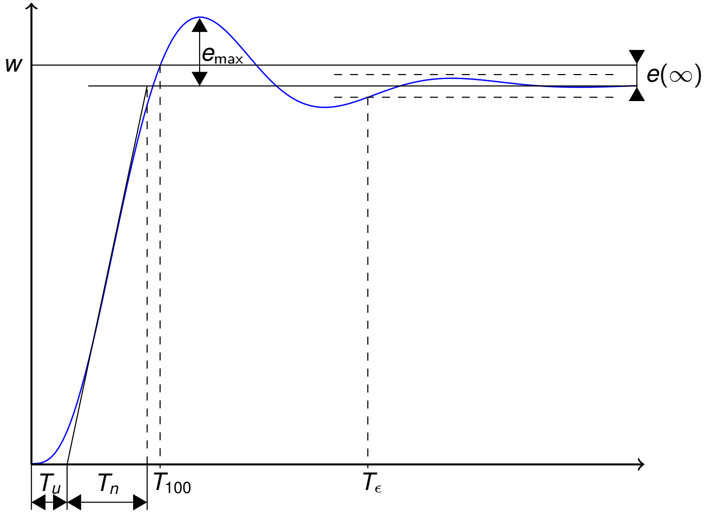
Second Order System – Conclusions

- All time indices are inversely proportional to ω_0.
- Integral indices are inversely proportional to ω_0^2.
- Maximum overshoot depends on ζ only and decreases with increasing ζ.

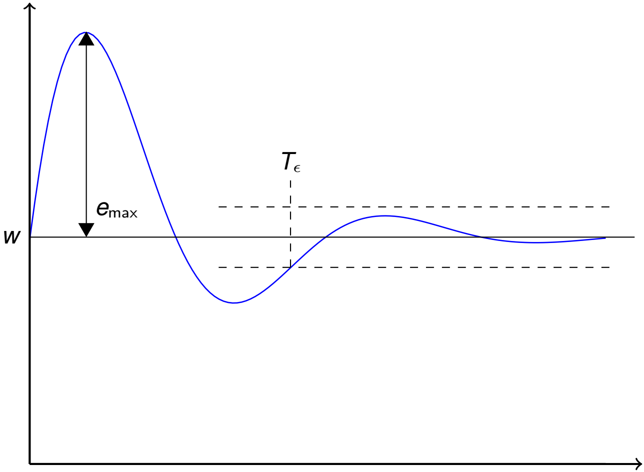
Poles and Zeros

- Poles: aperiodicity, stability
- Poles: slow down the response
- Effect of dominant poles
- Zeros: (non) minimum phase
- Zeros: speed up the response
- Effect of dominant zeros

Step Response – Tracking



Step Response – Regulation



Indices

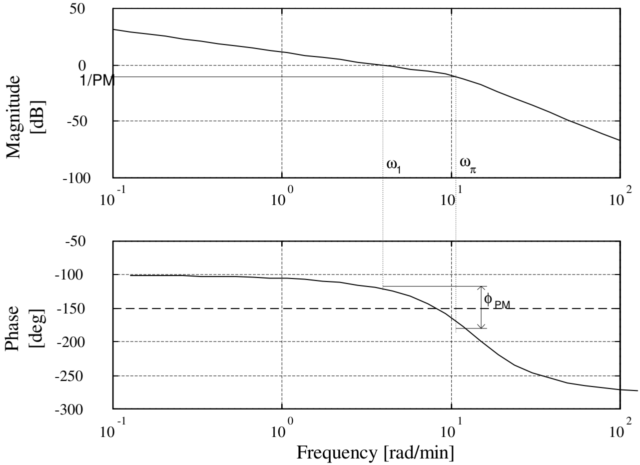
- Maximum overshoot $e_{\max}$. The recommended value: less than 25%.
- Peak time
- Settling time T_ϵ
- Damping ratio. The value of 0.3 or less can be suitable.
- Rise time T_{100}, T_{90}, $T_{10,90}$
- Dominant time constant T_{DOM}
- Steady-state control error $e(\infty)$

Integral Cost Functions

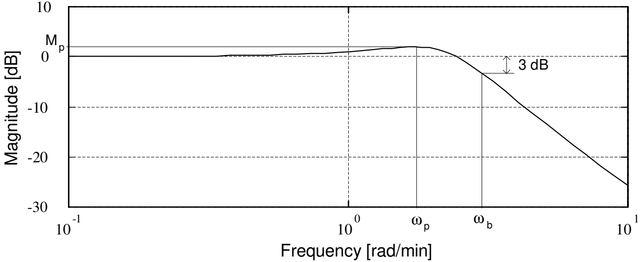
$$I = \int_0^\infty f[e(t)]\mathrm{d}t$$

- $f = e(t)$: IE = integral of error
- $f = |e(t)|$: IAE = integral absolute value of error
- $f = |e(t)|t$: ITAE = integral time multiplied absolute value of error
- $f = e^2(t)$: ISE = integral squared value of error
- $f = e^2(t) + \phi u^2(t)$: LQ - linear quadratic

Gain, Phase Margins



Bandwith


10
M p
0
-10
-20
-30
Magnitude [dB]
3 dB
10 -1
10 0
ω p
ω b
10 1
Frequency [rad/min]

Indices

Open loop:

- Gain margin, $GM_w \in (-12, -20)\,\text{dB}$, $GM_d \in (-4, -9)\,\text{dB}$
- Phase crossover frequency (ω_π, ω_p)
- Phase margin, $w : 40^\circ - 60^\circ$, $d : 20^\circ - 50^\circ$
- Gain crossover frequency (ω_1, ω_g)

Closed loop:

- Bandwidth ω_b (larger ω_b means faster response)
- Cutoff rate (separation signals from noise)
- Resonant peak $1.1 - 1.5$ (indication of relative stability)
- Resonant frequency

Second Order System – Frequency (Closed-Loop)

Resonant frequency:

$$\omega_r = \omega_0\sqrt{1-2\zeta^2}, \qquad |G(j\omega_r)| = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

Bandwidth ($|G(j\omega_b)| = 1/\sqrt{2}|G(0)|$):

$$\omega_b = \omega_0\sqrt{1-2\zeta^2+\sqrt{1+(1-2\zeta^2)^2}}$$

$$\tan\varphi_b = \frac{2\zeta\sqrt{1-2\zeta^2+\sqrt{1+(1-2\zeta^2)^2}}}{2\zeta^2-\sqrt{1+(1-2\zeta^2)^2}}$$

Useful approximations:

$$\frac{\omega_b}{\omega_0} \approx 1.8-1.1\zeta, \qquad \zeta\in(0.3,0.8)$$

$$|\varphi_b| \approx \pi-2.23\zeta, \qquad \zeta\in(0,1)$$

$$\omega_b T_{50} \approx 2.3, \qquad \zeta\in(0.3,0.8)$$

Second Order System – Frequency (Open-Loop)

Margins:

$$\frac{\omega_g}{\omega_0} = \sqrt{\sqrt{4\zeta^4 + 1} - 2\zeta^2}$$

$$\varphi_p = \arctan\left(2\zeta\frac{\omega_0}{\omega_g}\right)$$

Useful approximations:

$$\omega_g T_{50} \approx 1.5 - \frac{e_{\max}[\%]}{250}$$

$$70 \approx \varphi_p[^\circ] + e_{\max}[\%]$$

$$\zeta \approx \frac{\varphi_p}{100[^\circ]}, \quad \zeta \in (0.0, 0.6)$$

Butterworth polynomials

$$B_n(s) = \prod_{k=1}^{\frac{n}{2}} \left[s^2 - 2s\cos\left(\frac{2k+n-1}{2n}\pi\right) + 1 \right] \qquad n = \text{even}$$

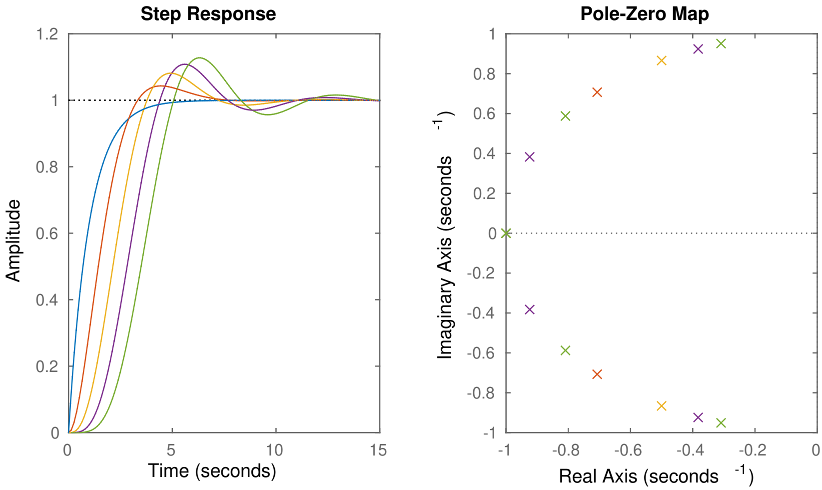
$$B_n(s) = (s+1)B_{n-1}(s) \qquad n = \text{odd}$$

Scaled:

$$B_1(s) = (Ts) + 1$$
$$B_2(s) = (Ts)^2 + 1.4142(Ts) + 1$$
$$B_3(s) = (Ts)^3 + 2(Ts)^2 + 2(Ts) + 1$$
$$B_4(s) = (Ts)^4 + 2.6131(Ts)^3 + 3.4142(Ts)^2 + 2.6131(Ts) + 1$$

```
n=3; T=1;
[b,a] = butter(n,1/T,'s'); g=tf(b,a);
step(g); pzmap(g)
```

Butterworth polynomials



Other Standard Polynomials

- Binomial form with a real stable pole $s = -\omega_0$ with mutiplicity of n

$$s + \omega_0$$
$$s^2 + 2\omega_0 s + \omega_0^2$$
$$s^3 + 3\omega_0 s^2 + 3\omega_0^2 s + \omega_0^3$$
$$s^4 + 4\omega_0 s^3 + 6\omega_0^2 s^2 + 4\omega_0^3 s + \omega_0^4$$

- Minimal $t_{5\%}$ – the fastest transient reponse with maximum overshoot of 5%.

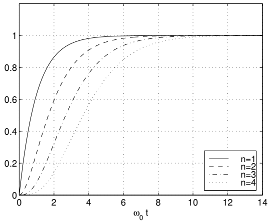
$$s + \omega_0$$
$$s^2 + 1.4\omega_0 s + \omega_0^2$$
$$s^3 + 1.55\omega_0 s^2 + 2.10\omega_0^2 s + \omega_0^3$$
$$s^4 + 1.60\omega_0 s^3 + 3.15\omega_0^2 s^2 + 2.45\omega_0^3 s + \omega_0^4$$

- Minimum of ITAE cost function.

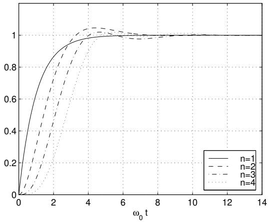
$$s + \omega_0$$
$$s^2 + 1.4\omega_0 s + \omega_0^2$$
$$s^3 + 1.75\omega_0 s^2 + 2.15\omega_0^2 s + \omega_0^3$$
$$s^4 + 2.1\omega_0 s^3 + 3.4\omega_0^2 s^2 + 2.7\omega_0^3 s + \omega_0^4$$

Standard Polynomials

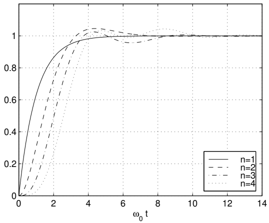
Binomial form



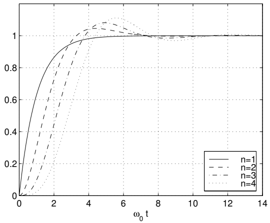
Min ITAE



Min $t_{5\%}$



Butterworth form



Closed-loop System – Poles, Zeros

- Feedback controller does not move plant zeros
- When control is expensive: place dominant closed-loop poles at stable plant poles and at mirrors of unstable plant poles
- When control is cheap: place dominant closed-loop poles at stable plant zeros and at mirrors of unstable plant zeros. Place other poles with Butterworth pattern.