Combined State Feedback/Observer

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TAR 1

Contents

Combined State Feedback and Observer

How does it affect stability if feedback control uses estimate of state $\hat{\boldsymbol{x}}$ instead of $\boldsymbol{x}$?

$\boldsymbol{u} = -\boldsymbol{K}_c\hat{\boldsymbol{x}} + \tilde{\boldsymbol{w}}$	tracking with observer
$\boldsymbol{e} = \boldsymbol{x} - \hat{\boldsymbol{x}}$	
$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$	controlled system
$\dot{\boldsymbol{e}} = (\boldsymbol{A} - \boldsymbol{K}_f\boldsymbol{C})\boldsymbol{e}$	observer error dynamics

Separation Principle

Combined dynamics

$$
\begin{aligned}
\dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} - \boldsymbol{B}\boldsymbol{K}_c\boldsymbol{x} + \boldsymbol{B}\boldsymbol{K}_c\boldsymbol{x} - \boldsymbol{B}\boldsymbol{K}_c\hat{\boldsymbol{x}} + \boldsymbol{B}\tilde{\boldsymbol{w}} \\
\dot{\boldsymbol{e}} &= (\boldsymbol{A} - \boldsymbol{K}_f\boldsymbol{C})\boldsymbol{e} \\
\begin{pmatrix} \dot{\boldsymbol{x}} \\ \dot{\boldsymbol{e}} \end{pmatrix} &= \begin{pmatrix} \boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}_c & \boldsymbol{B}\boldsymbol{K}_c \\ \boldsymbol{0} & \boldsymbol{A} - \boldsymbol{K}_f\boldsymbol{C} \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{e} \end{pmatrix} + \begin{pmatrix} \boldsymbol{B} \\ \boldsymbol{0} \end{pmatrix} \tilde{\boldsymbol{w}}
\end{aligned}
$$

- Closed-loop eigenvalues consist of eigenvalues of matrix $\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}_c$ (state feedback without observer) and of eigenvalues of matrix $\boldsymbol{A} - \boldsymbol{K}_f\boldsymbol{C}$ (observer dynamics).
- State feedback can be designed independently on the fact whether states are measurable or not.

Choice of Poles

Feedback:

- Poles according to performance specifications. Real dominant pole or couple of complex dominant poles should be faster than the corresponding ones of the controlled process. Other poles about 10 times faster.
- See the closed loop response and control actions. Change speed as needed.

Observer – should be faster than the controller so that control works with correct state estimates:

- Butterworth design. Time constant faster (2-6 times) than the dominant feedback pole.
- If there is significant noise in estimates, slow down the observer.

Linear Example

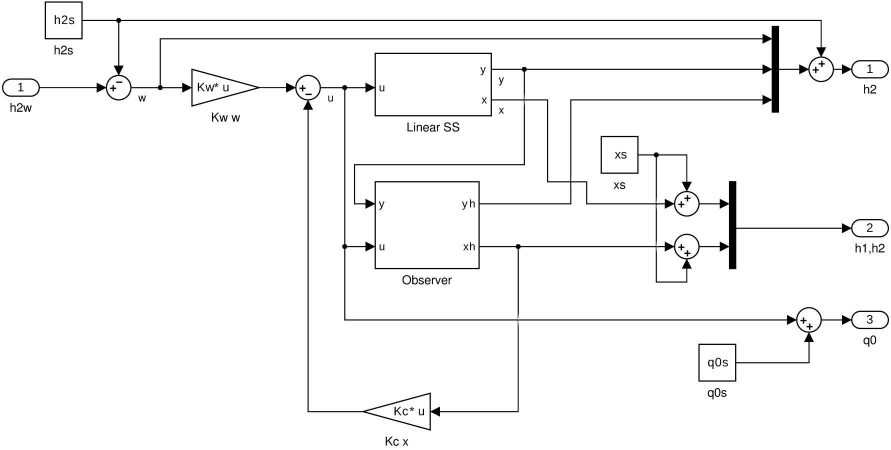
- Linearised two tank system with 2 states at $q_0^s = 0.9$, x_2 is measured.
- Full order observer with time constant $T_{63} = 1$:

```
T = 1; Tf = T/2;
Bf = [Tf^2 1.4142*Tf 1];
L  = place(A', C', roots(Bf));  Kf0  = L';
```

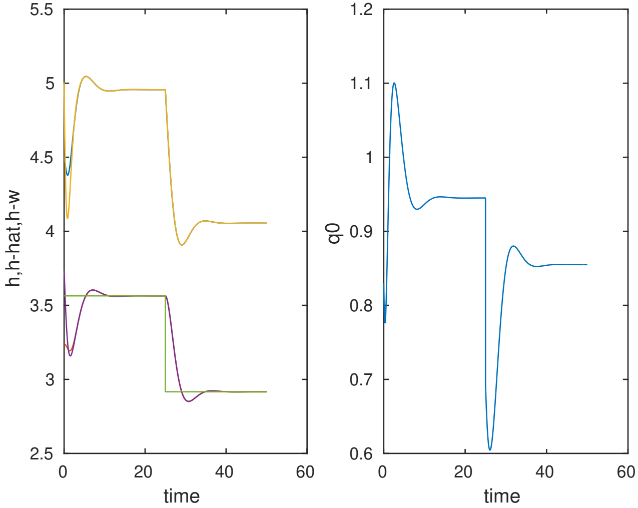
- State feedback controller with overshoot 10%, settling time 10.

```
sig = 0.1; teps = 10;
xi = abs(log(sig))/sqrt(pi^2+(log(sig))^2);
om0 = 4/(teps*xi);
pp = [1 2*xi*om0 om0^2];
Kc = place(A,B,roots(pp));
Kw = -1/(C*inv(A-B*Kc)*B);
```

Linear Example /2 – Simulink Scheme



Linear Example – Results



Combined Dist. Observer and Integral FB Control

Design observer for the process with states $\boldsymbol{x}, \boldsymbol{x}_d$:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}(\boldsymbol{u} + \boldsymbol{d}), \qquad \boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}$$

$$\dot{\boldsymbol{x}}_d = \boldsymbol{A}_d\boldsymbol{x}_d, \qquad \boldsymbol{d} = \boldsymbol{C}_d\boldsymbol{x}_d$$

Matrices:

$$\boldsymbol{A}_o = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B}\boldsymbol{C}_d \\ \boldsymbol{0} & \boldsymbol{A}_d \end{pmatrix}, \quad \boldsymbol{B}_o = \begin{pmatrix} \boldsymbol{B} \\ \boldsymbol{0} \end{pmatrix}, \quad \boldsymbol{C}_o = \begin{pmatrix} \boldsymbol{C} & \boldsymbol{0} \end{pmatrix}.$$

Observer:

$$\boldsymbol{K}_f^T = \mathrm{STAB}(\boldsymbol{A}_o^T, \boldsymbol{C}_o^T)$$

$$\dot{\tilde{\boldsymbol{x}}} = \boldsymbol{A}_o\tilde{\boldsymbol{x}} + \boldsymbol{B}_o\boldsymbol{u} + \boldsymbol{K}_f(\boldsymbol{y} - \boldsymbol{C}_o\tilde{\boldsymbol{x}})$$

$$\tilde{\boldsymbol{x}} = \begin{pmatrix} \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{x}}_d \end{pmatrix}$$

Combined Dist. Observer and Integral FB Control /2

Design state-space feedback controller for process with integrators:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}(\boldsymbol{u} + \boldsymbol{d}), \quad \boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}$$
$$\dot{\boldsymbol{x}}_i = \boldsymbol{w} - \boldsymbol{C}\boldsymbol{x}$$

Matrices:

$$\boldsymbol{A}_c = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{0} \\ -\boldsymbol{C} & \boldsymbol{0} \end{pmatrix}, \quad \boldsymbol{B}_c = \begin{pmatrix} \boldsymbol{B} \\ \boldsymbol{0} \end{pmatrix}$$

Controller:

$$\begin{pmatrix} \boldsymbol{K}_{c0} \\ \boldsymbol{K}_{ci} \end{pmatrix} = \mathrm{STAB}(\boldsymbol{A}_c, \boldsymbol{B}_c)$$
$$\boldsymbol{u} = -\boldsymbol{K}_{c0}\hat{\boldsymbol{x}} - \boldsymbol{K}_{ci}\boldsymbol{x}_i - \boldsymbol{C}_d\hat{\boldsymbol{x}}_d$$
$$= -\boldsymbol{K}_{c0}\hat{\boldsymbol{x}} - \boldsymbol{K}_{ci}\int \boldsymbol{e}\,\mathrm{d}t - \hat{\boldsymbol{d}}$$

Nonlinear Example

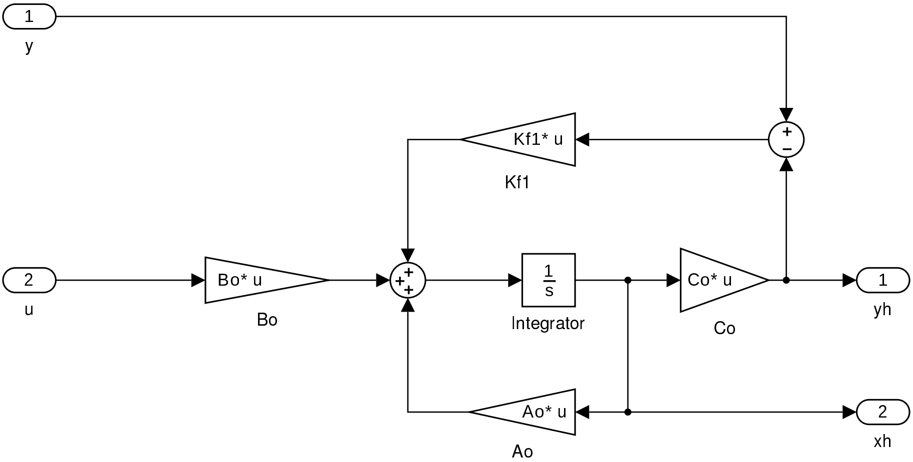
- Nonlinear two tank system with 2 states at $q_0^s = 0.9$, x_2 is measured with some noise.
- Full order observer for constant disturbances with time constant $T_{63} = 1$:

```
Ad = 0; Cd = 1; Ao = [A, B*Cd; zeros(1, 2) Ad];
Bo = [B; 0]; Co = [C, 0];
Tf = T/3; Bf = [Tf^3 2*Tf^2 2*Tf 1];
L  = place(Ao', Co', roots(Bf)); Kf1  = L';
```

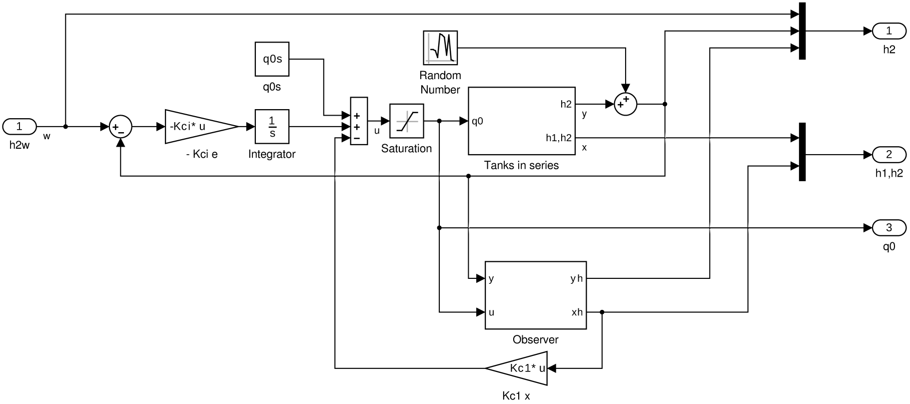
- State feedback controller with integral action for constant setpoints, overshoot 10%, settling time 10.

```
Af = [A zeros(2,1); -C zeros(1,1)]; Bf = [B; 0];
pp1 = conv(pp, [1 10]);
K = place(Af,Bf,roots(pp1));
Kc0 = K(1:2); Kci = K(3); Kc1 = [K(1) K(2) 1];
```

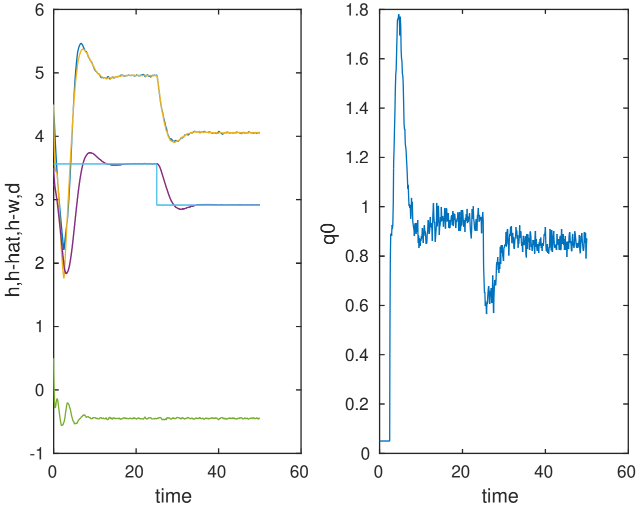
Nonlinear Example /2a – Simulink Scheme



Nonlinear Example /2b – Simulink Scheme



Nonlinear Example – Results



Setup

Controlled SISO system, n states, completely controllable and observable:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u, \quad y = \boldsymbol{C}\boldsymbol{x}$$

$$\frac{b(s)}{a(s)} = \boldsymbol{C}\left(s\boldsymbol{I} - \boldsymbol{A}\right)^{-1}\boldsymbol{B}, \quad \text{GCD}(a, b) = 1$$

$$\frac{\boldsymbol{B}_{Rs}(s)}{a(s)} = \left(s\boldsymbol{I} - \boldsymbol{A}\right)^{-1}\boldsymbol{B}, \quad \text{GCD}(a, \boldsymbol{B}_{Rs}) = 1$$

$$y = b(s)x_p, \quad u = a(s)x_p, \quad \boldsymbol{x} = \boldsymbol{B}_{Rs}(s)x_p$$

Control is state feedback:

$$u = -\boldsymbol{K}_c\hat{\boldsymbol{x}} + \tilde{w}$$

Observer:

$$\dot{\hat{\boldsymbol{x}}} = \boldsymbol{A}\hat{\boldsymbol{x}} + \boldsymbol{B}u + \boldsymbol{K}_f\left(y - \boldsymbol{C}\hat{\boldsymbol{x}}\right)$$

Task: to derive the equivalent input–output representation

Derivation /1 – State feedback

$$
\begin{aligned}
(s\boldsymbol{I} - \boldsymbol{A})\boldsymbol{B}_{Rs}(s) &= \boldsymbol{B}a(s) \\
(s\boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}_c))\,\boldsymbol{B}_{Rs}(s) &= \boldsymbol{B}\,(a(s) + \boldsymbol{K}_c\boldsymbol{B}_{Rs}(s)) \\
\frac{\boldsymbol{B}_{Rs}(s)}{a(s) + \boldsymbol{K}_c\boldsymbol{B}_{Rs}(s)} &= \frac{\operatorname{adj}(s\boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}_c))}{\det(s\boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}_c))}\boldsymbol{B}
\end{aligned}
$$

Let $f(s)$ be the characteristic polynomial of the closed-loop state feedback $\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}_c$.

$$
f(s) = \det(s\boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}_c))
$$

The feedback gain can be calculated from

$$
\begin{aligned}
a(s) + \boldsymbol{K}_c\boldsymbol{B}_{Rs}(s) &= f(s) \\
a(s) + k(s) &= f(s), \quad k(s) = \boldsymbol{K}_c\boldsymbol{B}_{Rs}(s)
\end{aligned}
$$

Derivation /2 – Observer

$$
\begin{aligned}
s\hat{\boldsymbol{x}} &= (\boldsymbol{A} - \boldsymbol{K}_f\boldsymbol{C})\,\hat{\boldsymbol{x}} + \boldsymbol{B}u + \boldsymbol{K}_f y \\
\hat{\boldsymbol{x}} &= (s\boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{K}_f\boldsymbol{C}))^{-1}\,\boldsymbol{B}u + (s\boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{K}_f\boldsymbol{C}))^{-1}\,\boldsymbol{K}_f y \\
\hat{\boldsymbol{x}} &= \frac{\boldsymbol{T}_u(s)}{o(s)}u + \frac{\boldsymbol{T}_y(s)}{o(s)}y \\
\boldsymbol{T}_u(s) &= \operatorname{adj}(s\boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{K}_f\boldsymbol{C}))\,\boldsymbol{B} \\
\boldsymbol{T}_y(s) &= \operatorname{adj}(s\boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{K}_f\boldsymbol{C}))\,\boldsymbol{K}_f \\
o(s) &= \det(sI - (\boldsymbol{A} - \boldsymbol{K}_f\boldsymbol{C})) \\
\tilde{u} &= \boldsymbol{K}_c\hat{\boldsymbol{x}} = \frac{r(s)}{o(s)}u + \frac{q(s)}{o(s)}y, \quad r(s) = \boldsymbol{K}_c\boldsymbol{T}_u(s), \quad q(s) = \boldsymbol{K}_c\boldsymbol{T}_y(s)
\end{aligned}
$$

Derivation /3 – Partial State

$$
\begin{aligned}
y = b(s)x_p, \quad u = a(s)x_p, \quad \boldsymbol{x} = \boldsymbol{B}_{Rs}(s)x_p \\
\boldsymbol{K}_c\boldsymbol{x} &= \boldsymbol{K}_c\boldsymbol{B}_{Rs}(s)x_p = k(s)x_p \\
\boldsymbol{K}_c\boldsymbol{x} - \boldsymbol{K}_c\hat{\boldsymbol{x}} &= k(s)x_p - \frac{r(s)}{o(s)}u - \frac{q(s)}{o(s)}y \\
o(s)\boldsymbol{K}_c\left(\boldsymbol{x} - \hat{\boldsymbol{x}}\right) &= o(s)k(s)x_p - r(s)u - q(s)y \\
&= \left(o(s)k(s) - r(s)a(s) + q(s)b(s)\right)x_p
\end{aligned}
$$

Stable observer zeroes state estimation error

$$
r(s)a(s) + q(s)b(s) = o(s)k(s)
$$

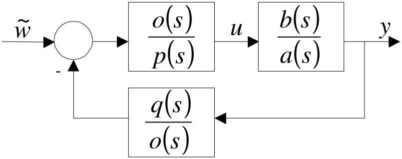
Let $p(s) = o(s) + r(s)$. Then

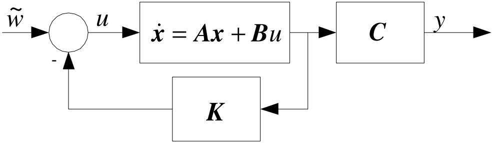
$$
\frac{p(s)}{o(s)}u = -\frac{q(s)}{o(s)}y + \tilde{w}
$$

$$
a(s)p(s) + b(s)q(s) = o(s)f(s)
$$

Controller from Diophantine equation

Equivalence





$$y = \frac{b(s)o(s)}{a(s)p(s) + b(s)q(s)}\tilde{w}$$

$$y = \frac{b(s)}{f(s)}\tilde{w}$$

$$y = \boldsymbol{C}\left(s\boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}_c)\right)^{-1}\boldsymbol{B}\tilde{w}$$

$$y = \boldsymbol{C}\frac{\operatorname{adj}\left(s\boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}_c)\right)}{\det\left(s\boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}_c)\right)}\boldsymbol{B}\tilde{w}$$

$$y = \frac{b(s)}{f(s)}\tilde{w}$$

One-degree-of-freedom Controller

Consider setpoint

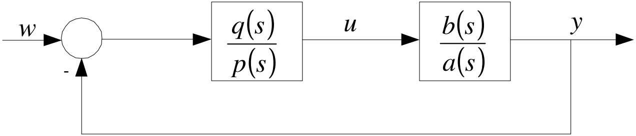
$$\tilde{w} = \frac{q(s)}{o(s)} w$$

Control

$$u = \frac{q(s)}{p(s)} (w - y)$$

Output

$$y = \frac{b(s)q(s)}{o(s)f(s)} w$$



Polynomial Pole Placement

1. Given are state-space matrices of the controlled process $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and gain matrices $\boldsymbol{K}_c$ and $\boldsymbol{K}_f$.
2. Polynomials $a(s)$ and $b(s)$ are calculated from

$$\boldsymbol{C}\left(s\boldsymbol{I} - \boldsymbol{A}\right)^{-1}\boldsymbol{B} = \frac{b(s)}{a(s)}$$

3. Controller polynomials $p(s)$ and $q(s)$ are found from Diophantine equation

$$a(s)p(s) + b(s)q(s) = o(s)f(s)$$

where

$$o(s) = \det\left(s\boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{K}_f\boldsymbol{C})\right)$$
$$f(s) = \det\left(s\boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}_c)\right)$$

Polynomial $c(s) = o(s)f(s)$ has the degree $2n$ (full order observer) or $2n - 1$ (reduced order observer).

Example

- Given are state-space matrices of the controlled process $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and gain matrices $\boldsymbol{K}_c$ and $\boldsymbol{K}_f$.

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{9} & -\frac{6}{9} \end{pmatrix}, \qquad \boldsymbol{B} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \qquad \boldsymbol{C} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$\boldsymbol{K}_c = \begin{pmatrix} 1 & 1 \end{pmatrix}, \qquad \boldsymbol{K}_f = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

- Polynomials $a(s)$ and $b(s)$ are calculated from

$$\boldsymbol{C}\,(s\boldsymbol{I} - \boldsymbol{A})^{-1}\,\boldsymbol{B} = \frac{b(s)}{a(s)} = \frac{2}{s^2 + \frac{6}{9}s + \frac{1}{9}}$$

- Observer and feedback polynomials

$$o(s) = \det\left(s\boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{K}_f\boldsymbol{C})\right) = s^2 + \frac{24}{9}s + \frac{22}{9}$$

$$f(s) = \det\left(s\boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}_c)\right) = s^2 + \frac{24}{9}s + \frac{19}{9}$$

Example

- Controller polynomials $p(s)$ and $q(s)$ are found from Diophantine equation

$$a(s)p(s) + b(s)q(s) = o(s)f(s)$$

with degrees $\deg(q) = 1$, $\deg(p) = 2$ (we minimise $\deg(q)$)

$$p(s) = s^2 + \frac{14}{3}s + \frac{76}{9}$$
$$q(s) = 3s + \frac{19}{9}$$

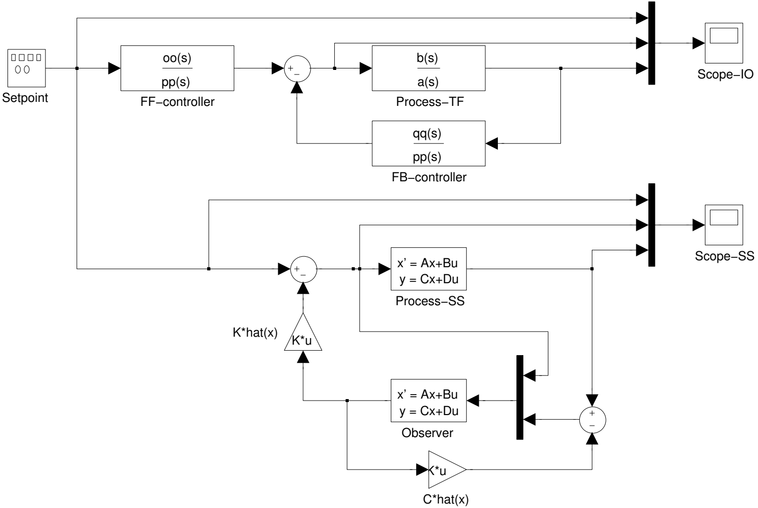
- Control law

$$u = -\boldsymbol{K}_c\hat{\boldsymbol{x}} + \tilde{w}$$
$$\dot{\hat{\boldsymbol{x}}} = \boldsymbol{A}\hat{\boldsymbol{x}} + \boldsymbol{B}u + \boldsymbol{K}_f(y - \boldsymbol{C}\hat{\boldsymbol{x}})$$
$$= \boldsymbol{A}\hat{\boldsymbol{x}} + \begin{pmatrix}\boldsymbol{B} & \boldsymbol{K}_f\end{pmatrix}\begin{pmatrix} u \\ y - \boldsymbol{C}\hat{\boldsymbol{x}}\end{pmatrix}$$

$$u = \frac{o(s)}{p(s)}\tilde{w} - \frac{q(s)}{p(s)}y$$
$$= \frac{s^2 + \frac{24}{9}s + \frac{22}{9}}{s^2 + \frac{14}{3}s + \frac{76}{9}}\tilde{w} - \frac{3s + \frac{19}{9}}{s^2 + \frac{14}{3}s + \frac{76}{9}}y$$

Example – examplesim.mdl

example.m, examplesim.mdl



Example – example.m

```
A = [0 1; -1/9 -6/9]; B = [0;2]; C=[1 0];
system = ss(A,B,C,[]); [b,a]=tfdata(system,'v');
Kc=[1 1]; Kf=[2;1];

syms s
f = det(s*eye(2)-(A-B*Kc)); o = det(s*eye(2)-(A-Kf*C));
aa = s^2+a(2)*s+a(3); bb = b(3);

syms p1 p0 q1 q0 real
p = s^2+p1*s+p0; q = q1*s+q0;
collect(expand(aa*p+bb*q-o*f),s)

%p1 = 14/3; p0=76/9; q0=19/9; q1=3;
pp = [1 14/3 76/9]; qq = [3 19/9]; oo = [1 24/9 22/9];
```